

# How high can Baumgartner's $\mathcal{I}$ -ultrafilters lie in the P-hierarchy ?

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## Abstract

Under CH we prove that for any tall ideal  $\mathcal{I}$  on  $\omega$  and for any ordinal  $\gamma \leq \omega_1$  there is an  $\mathcal{I}$ -ultrafilter (in the sense of Baumgartner), which belongs to the class  $\mathcal{P}_\gamma$  of P-hierarchy of ultrafilters. Since the class of  $\mathcal{P}_2$  ultrafilters coincides with a class of P-points, our result generalizes theorem of Flašková, which states that there are  $\mathcal{I}$ -ultrafilters which are not P-points.

## 1 Introduction

Baumgartner in the article *Ultrafilters on  $\omega$*  ([1]) introduced a notion of  $\mathcal{I}$ -ultrafilters:

Let  $\mathcal{I}$  be an ideal on  $\omega$ . A filter on  $\omega$  is an  $\mathcal{I}$ -ultrafilter, if and only if, for every function  $f \in \omega^\omega$  there is a set  $U \in \mathcal{u}$  such that  $f[U] \in \mathcal{I}$ .

This kind of ultrafilters was studied by large group of mathematicians. We shall mention only the most important papers in this subject from our point of view: J. Brendle [3], C. Laflamme [17], Shelah [20], [21], Błaszczyk [2]. The theory of  $\mathcal{I}$ -ultrafilters was developed by Flašková in a series of articles and in her Ph.D thesis [10].

In [10] Flašková proved under CH that for every tall P-ideal  $\mathcal{I}$  that contains all singletons, there is an  $\mathcal{I}$ -ultrafilter, which is not a P-point. Later she succeeded to replace the assumption of CH by  $\mathfrak{p} = \mathfrak{c}$  [9].

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Ultrafilters on  $\omega$  may be classified with respect to sequential contours of different ranks, that is, iterations of the Fréchet filter by contour operations. This way an  $\omega_1$ -sequence  $\{\mathcal{P}_\alpha\}_{1 \leq \alpha \leq \omega_1}$  of pairwise disjoint classes of ultrafilters - the P-hierarchy - is obtained, where P-points correspond to the class  $\mathcal{P}_2$ , allowing us to look at the P-hierarchy as the extension of P-points. The following theorem was proved by Starosolski, see [23] Proposition 2.1:

**Proposition 1.1.** *An ultrafilter  $u$  is a P-point if and only if  $u$  belongs to the class  $\mathcal{P}_\infty$  in P-hierarchy.*

All necessary information about P-hierarchy may be found in [23]. For additional information regarding sequential cascades and contours a look at [7], [8], [6], [22] is recommended. However we shall repeat the most important definitions and conventions below.

Since P-point correspond to  $\mathcal{P}_2$  ultrafilter in P-hierarchy of ultrafilters (more about P-hierarchy one can find below), it would interesting to know to which classes of P-hierarchy can belong  $\mathcal{I}$ -ultrafilters. In this paper we shall show that it can be any class  $\mathcal{P}_\alpha$ . Let us introduce all necessary definitions and tools.

The set of natural numbers (finite ordinal numbers) we denote  $\omega$ . The filter considered in this paper will be defined on infinite countable set (except one indicated case) . This will be usually a set  $\max V$  of maximal elements of a cascade  $V$  (see definition of cascade below) and we will often identify it with  $\omega$  without indication. The following convention we be applied without mentioning it:

*Convention:* If  $u$  is a filter on  $A \subset B$ , then we identify  $u$  with the filter on  $B$  for which  $u$  is a filter-base. If  $\mathcal{F}$  is a filter base, then by  $\langle \mathcal{F} \rangle$  we denote a filter generated by  $\mathcal{F}$ .

The *cascade* is a tree  $V$  without infinite branches and with a least element  $\emptyset_V$ . A cascade is *sequential* if for each non-maximal element of  $V$  ( $v \in V \setminus \max V$ ) the set  $v^{+V}$  of immediate successors of  $v$  (in  $V$ ) is countably infinite. We write  $v^+$  instead of  $v^{+W}$  if it is known in which cascade the successors of  $v$  are considered. If  $v \in V \setminus \max V$ , then the set  $v^+$  (if infinite) may be endowed with an order of the type  $\omega$ , and then by  $(v_n)_{n \in \omega}$  we denote the sequence of elements of  $v^+$ , and by  $v_{nW}$  - the  $n$ -th element of  $v^{+W}$ .

The *rank* of  $v \in V$  ( $r_V(v)$  or  $r(v)$ ) is defined inductively as follows:  $r(v) = 0$  if  $v \in \max V$ , and otherwise  $r(v)$  is the least ordinal greater than the ranks of all immediate successors of  $v$ . The rank  $r(V)$  of the cascade  $V$

is, by definition, the rank of  $\emptyset_V$ . If it is possible to order all sets  $v^+$  (for  $v \in V \setminus \max V$ ) so that for each  $v \in V \setminus \max V$  the sequence  $(r(v_n)_{n < \omega})$  is non-decreasing, then the cascade  $V$  is *monotone*, and we fix such an order on  $V$  without indication.

For  $v \in V$  we denote by  $v^\uparrow$  a subcascade of  $V$  built by  $v$  and all successors of  $v$ . We write  $v^\uparrow$  instead of  $v^\uparrow^V$  if we know in which cascade the subcascade is included.

If  $\mathbb{F} = \{\mathcal{F}_s : s \in S\}$  is a family of filters on  $X$  and if  $\mathcal{G}$  is a filter on  $S$ , then the *contour* of  $\{\mathcal{F}_s\}$  along  $\mathcal{G}$  is defined by

$$\int_{\mathcal{G}} \mathbb{F} = \int_{\mathcal{G}} \{\mathcal{F}_s : s \in S\} = \bigcup_{G \in \mathcal{G}} \bigcap_{s \in G} \mathcal{F}_s.$$

Such a construction has been used by many authors ([11], [12], [13]) and is also known as a sum (or as a limit) of filters. On the sequential cascade, we consider the finest topology such that for all but the maximal elements  $v$  of  $V$ , the co-finite filter on the set  $v^{+V}$  converges to  $v$ . For the sequential cascade  $V$  we define the *contour* of  $V$  (we write  $\int V$ ) as the trace on  $\max V$  of the neighborhood filter of  $\emptyset_V$  (the *trace* of a filter  $u$  on a set  $A$  is the family of intersections of elements of  $u$  with  $A$ ). Similar filters were considered in [14], [15], [5]. Let  $V$  be a monotone sequential cascade and let  $u = \int V$ . Then a *rank*  $r(u)$  of  $u$  is, by definition, the rank of  $V$ .

It was shown in [8] that if  $\int V = \int W$ , then  $r(V) = r(W)$ .

We shall say that a set  $F$  *meshes* a contour  $\mathcal{V}$  ( $F \# \mathcal{V}$ ) if and only if  $\mathcal{V} \cup \{F\}$  has finite intersection property and can be extended to a filter. If  $\omega \setminus F \in \mathcal{V}$ , then we say that  $F$  is *residual* with respect to  $\mathcal{V}$ .

Let us define  $\mathcal{P}_\alpha$  for  $1 \leq \alpha < \omega_1$  on  $\beta\omega$  (see [23]) as follows:  $u \in \mathcal{P}_\alpha$  if there is no monotone sequential contour  $C_\alpha$  of rank  $\alpha$  such that  $C_\alpha \subset u$ , and for each  $\beta$  in the range  $1 \leq \beta < \alpha$  there exists a monotone sequential contour  $C_\beta$  of rank  $\beta$  such that  $C_\beta \subset u$ . Moreover, if for each  $\alpha < \omega_1$  there exists a monotone sequential contour  $C_\alpha$  of rank  $\alpha$  such that  $C_\alpha \subset u$ , then we write  $u \in \mathcal{P}_{\omega_1}$ .

Let us consider a monotone cascade  $V$  and a monotone sequential cascade  $W$ . We will say that  $W$  is a sequential extension of  $V$  if:

- 1)  $V$  is a subcascade of cascade  $W$ ,

- 2) if  $v^{+V}$  is infinite, then  $v^{+V} = v^{+W}$ ,
- 3)  $r_V(v) = r_W(v)$  for each  $v \in V$ .

Obviously, a monotone cascade may have many sequential extensions.

Notice that if  $W$  is a sequential extension of  $V$  and  $U \subset \max V$ , then  $U$  is residual for  $V$  if and only if  $U$  is residual for  $W$ .

It cannot be proven in ZFC that classes  $\mathcal{P}_\alpha$  are nonempty. The following theorem was proved in [23] Theorem 2.8:

**Theorem 1.2.** *The following statements are equivalent:*

- 1. *P-points exist,*
- 2.  *$\mathcal{P}_\alpha$  classes are non-empty for each countable successor  $\alpha$ ,*
- 3. *There exists a countable successor  $\alpha > 1$  such that the class  $\mathcal{P}_\alpha$  is non-empty.*

Starosolski has proved in [25] Theorem 6.7 that under CH every class  $\mathcal{P}_\alpha$  is nonempty.

**Theorem 1.3.** *Assuming CH every class  $\mathcal{P}_\alpha$  is nonempty*

The main theorem presented in this paper is on the one side an extension of Starosolski's result, but on the side based on it.

Let us consider another technical notion which one could call a "restriction of a cascade". Let  $V$  be a monotone sequential cascade and let a set  $H$  mesh the contour  $\int V$ . By  $V^{\downarrow H}$  we denote a biggest monotone sequential cascade such that  $V^{\downarrow H} \subset V$  and  $\max V^{\downarrow H} \subset H$ . It is easy to see that  $H \in \int V^{\downarrow H}$ .

At the end of introduction let us reformulate a definition of the contour in a special case when on each node of a cascade is given (fixed) arbitrary filter (not necessarily Frechet):

Fix a cascade  $V$ . Let  $\mathcal{G}(v)$  be a filter on  $v^+$  for every  $v \in V \setminus \max V$ . For  $v \in \max V$  let  $\mathcal{G}(v)$  be a trivial ultrafilter on a singleton  $\{v\}$  (we can treat it as principal ultrafilter on  $\max v$  according to convention we assumed). Thus we have defined a function  $v \mapsto \mathcal{G}(v)$ . We define contour of every sub-cascade  $v^\uparrow$  inductively with respect to rank of  $v$ :

$$\int^{\mathcal{G}} v^\uparrow = \{\{v\}\}$$

for  $v \in \max V$  (i.e.  $\int^{\mathcal{G}} v^\uparrow$  is just a trivial ultrafilter on singleton  $\{v\}$ ) ;

$$\int^{\mathcal{G}} v^\uparrow = \int_{\mathcal{G}(v)} \left\{ \int^{\mathcal{G}} w^\uparrow : w \in v^+ \right\}$$

for  $v \in V \setminus \max v$ .

## 2 Lemmas

The following lemmas will be used in the prove of a main theorem.

The first lemma is one of lemmas proved in [24] (see: Lemma 2.3 ):

**Lemma 2.1.** *Let  $\alpha < \omega_1$  be a limit ordinal and let  $(\mathcal{V}_n : n < \omega)$  be a sequence of monotone sequential contours such that  $r(\mathcal{V}_n) < r(\mathcal{V}_{n+1}) < \alpha$  for every  $n$  and that  $\bigcup_{n < \omega} \mathcal{V}_n$  has finite intersection property. Then there is no monotone sequential contour  $\mathcal{W}$  of rank  $\alpha$  such that  $\mathcal{W} \subset \langle \bigcup_{n < \omega} \mathcal{V}_n \rangle$ .*

As a corollary we get:

**Lemma 2.2.** *Let  $\alpha < \omega_1$  be a limit ordinal, let  $(\mathcal{V}_n)_{n < \omega}$  be an increasing (" $\subset$ ") sequence of monotone sequential contours, such that  $r(\mathcal{V}_n) < \alpha$  and let  $\mathcal{F}$  be a countable family of sets such that  $\bigcup_{n < \omega} \mathcal{V}_n \cup \mathcal{F}$  has finite intersection property. Then  $\langle \bigcup_{n < \omega} \mathcal{V}_n \cup \mathcal{F} \rangle$  do not contain any monotone sequential contour of rank  $\alpha$ .*

*Proof:* Assume that  $\mathcal{F}$  is finite. Let  $\mathcal{W}_n = \{U \cap \bigcap \mathcal{F} : U \in \mathcal{V}_n\}$ . It is easy to see that  $\mathcal{W}_n$  is monotone sequential contour of the same rank as  $\mathcal{V}_n$ . Consider a sequence  $(\mathcal{W}_n)$ . By Lemma 2.1 the union  $(\mathcal{W}_n)$  do not contains contour of rank  $\alpha$ .

Assume that  $\mathcal{F}$  is infinite. Order  $\mathcal{F}$  in  $\omega$  type, obtaining a sequence  $(F_n)_{n < \omega}$ . Next put

$$\mathcal{W}_n = \{U \cap \bigcap_{i \leq n} F_i : U \in \mathcal{V}_n\}.$$

Consider a sequence  $(\mathcal{W}_n : n < \omega)$  and use again Lemma 2.1 to show that the union  $(\mathcal{W}_n : n < \omega)$  do not contains contour of rank  $\alpha$ . ■

The following lemma is a straightforward extension of the claim contained in the proof of [9] Theorem 3.2. and since a proof is almost identical to the quoted one, we left it to the reader.

**Lemma 2.3.** *Let  $\mathcal{I}$  be a tall  $P$ -ideal that contains all singletons, let  $\{U_n : n < \omega\}$  be a pairwise disjoint sequence of subsets of  $\omega$ , let  $\{u_n : n < \omega\}$  be a sequence of  $\mathcal{I}$ -ultrafilters such that  $U_n \in u_n$ , finally let  $v$  be another one  $\mathcal{I}$ -ultrafilter. Then  $\int_v \{u_n : n < \omega\}$  is a  $\mathcal{I}$ -ultrafilter.*

As immediate consequence we get

**Lemma 2.4.** *If  $V$  is monotone sequential cascade,  $\mathcal{G}(v)$  is an  $P$ -point and  $\mathcal{I}$ -ultrafilter for each  $v \in V \setminus \max V$  and  $\mathcal{G}(v)$  is a trivial ultrafilter on a singleton  $\{v\}$  for  $v \in \max V$ , then  $\int^{\mathcal{G}} V$  is an  $\mathcal{I}$ -ultrafilter.*

Similar lemma we can formulate for ultrafilters from  $P$ -hierarchy instead of  $\mathcal{I}$ -ultrafilters, see [23] Theorem 2.5:

**Theorem 2.5.** *Let  $V$  be a monotone sequential cascade of rank  $\gamma$ , let  $G(v)$  be a principal ultrafilter on  $\{v\}$  for  $v \in \max V$ , and let  $G(v)$  be a  $P$ -point on  $v^+$  for  $v \in V \setminus \max V$ . Then  $\int^G V \in P_{\gamma+1}$ .*

**Corollary 2.6.** *If  $V$  is monotone sequential cascade,  $\mathcal{G}(v)$  is an ultrafilter from the class  $\mathcal{P}_\gamma$  for each  $v \in V \setminus \max V$  and  $\mathcal{G}(v)$  is a trivial ultrafilter on a singleton  $\{v\}$  for  $v \in \max V$ , then  $\int^{\mathcal{G}} V$  belongs to the class  $\mathcal{P}_\gamma$ .*

In above theorem and corollary we can identify principal ultrafilter on  $\{v\}$  with principal ultrafilter generated on  $\omega$  by  $v$ .

### 3 Main result

In this section we shall present main result of the paper.

**Theorem 3.1.** *(CH) Let  $\mathcal{I}$  be a tall  $P$ -ideal that contain all singletons, and let  $\gamma \leq \omega_1$  be an ordinal. Then there exists an  $\mathcal{I}$ -ultrafilter  $u$  which belongs to  $\mathcal{P}_\gamma$ .*

*Proof.* We shall split proof into five cases:  $\gamma = 1$ ,  $\gamma = 2$ ,  $\gamma > 2$  is a successor ordinal (the main step),  $\gamma < \omega_1$  is limit ordinal,  $\gamma = \omega_1$ .

**Step 0:**  $\gamma = 1$  is clear, image of singleton ( $\mathcal{P}_1$  is a class of principal ultrafilters) is a singleton, so belongs to  $\mathcal{I}$ .

**Step 1:** for  $\gamma = 2$ .

We order all contours of rank 2 and all functions  $\omega \rightarrow \omega$  in  $\omega_1$ -sequences  $(\mathcal{W}_\alpha)_{\alpha < \omega_1}$ ,  $(f_\alpha)_{\alpha < \omega_1}$  respectively. By transfinite induction, for  $\alpha < \omega_1$  we build countable generated filters  $\mathcal{F}_\alpha$  together with their decreasing basis  $(F_\alpha^n)_{n < \omega}$ , such that:

1.  $\mathcal{F}_0$  is a Frechet filter;
2. for each  $\alpha < \omega_1$  the sequence  $(F_\alpha^n)_{n < \omega}$  is strictly decreasing base of  $\mathcal{F}_\alpha$ ;

3.  $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$  for  $\alpha < \beta$ ;
4.  $\mathcal{F}_\alpha = \bigcup_{\beta < \alpha} \mathcal{F}_\beta$  for  $\alpha$  limit ordinal;
5. for each  $\alpha < \omega_1$  there is  $F \in \mathcal{F}_{\alpha+1}$  such that  $f_\alpha[F] \in \mathcal{I}$ ;
6. for each  $\alpha < \omega_1$  there is  $F \in \mathcal{F}_{\alpha+1}$  such that a complement of  $F$  belongs to  $\mathcal{W}_\alpha$ .

Suppose that  $\mathcal{F}_\alpha$  is already define, we will show how to build  $\mathcal{F}_{\alpha+1}$ . Since  $F_\alpha^n$  is strictly decreasing one can pick  $x_n \in F_\alpha^n \setminus F_\alpha^{n+1}$  for every  $n < \omega$ . Put  $T = \{x_n : n < \omega\}$ . The are two possibilities:

If  $f_\alpha[T]$  is finite then there is  $j \in f_\alpha[T]$  such that a preimage  $f_\alpha^{-1}[j]$  intersect infinite many of  $F_\alpha^n \setminus F_\alpha^{n+1}$ . In this case put  $G = f_\alpha^{-1}[j]$ .

If  $f_\alpha[T]$  is infinite, then since  $\mathcal{I}$  is tall there is  $I \in \mathcal{I}$  such that  $I \subset f_\alpha[T]$ . This time put  $G = f_\alpha^{-1}[I]$ .

Notice that  $\{F_\alpha^n : n < \omega\} \cup \{G_\alpha\}$  has finite intersection property and is countable. By last property there is no subbase of any monotone sequential contour of rank 2 that is contained in  $\{F_\alpha^n : n < \omega\} \cup \{G_\alpha\}$ . So there is a set  $A_\alpha$  such that its complement belongs to  $\mathcal{W}_\alpha$  and a family  $\{F_\alpha^n : n < \omega\} \cup \{G_\alpha, A_\alpha\}$  has finite intersection property. Order  $\{F_\alpha^n : n < \omega\} \cup \{G_\alpha\} \cup \{A_\alpha\}$  in  $\omega$  type, obtaining a sequence  $(\tilde{F}_{\alpha+1}^n : n < \omega)$ . Put  $F_{\alpha+1}^n = \bigcap_{m \leq n} \tilde{F}_{\alpha+1}^m$  to get decreasing sequence and let  $\mathcal{F}_{\alpha+1} = \langle \{F_{\alpha+1}^n : n < \omega\} \rangle$ .

Take any ultrafilter  $u$  that extends  $\bigcup_{\alpha < \omega_1} \mathcal{F}_\alpha$ . By condition 5)  $u$  is an  $\mathcal{I}$ -ultrafilter, by condition 6)  $u$  do not contain any monotone sequential contour of rank 2. Since by condition 1)  $u$  contains a Frechet filter it is not principal. Thus  $u$  is a P-point. (Note that on this step we do not use an assumption, that  $\mathcal{I}$  is a P-ideal.)

**Step 2:**  $\gamma$  is an arbitrary successor ordinal such that  $2 < \gamma < \omega_1$ . Let  $V$  ba an arbitrary monotone sequential cascade of rank  $\gamma - 1$ . Let  $V \in v \mapsto \mathcal{G}(v)$  be a function such that:

1)  $\mathcal{G}(v)$  is an P-point and  $\mathcal{I}$ -ultrafilter for each  $v \in V \setminus \max V$  (such ultrafilters exists by step 1)

2)  $\mathcal{G}(v)$  be a trivial ultrafilter on a singleton  $\{v\}$  for  $v \in \max V$ .

Lemma 2.5 guarantee that  $\int^\mathcal{G} V \in \mathcal{P}_\gamma$ . whilst Lemma 2.4 ensures us that  $\int^\mathcal{G} V$  is an  $\mathcal{I}$ -ultrafilter.

So we are done for successor  $\gamma$ .

**Step 3:** for limit  $\gamma < \omega_1$ . The proof in this case is base on the same idea as step 1, but it is more sophisticated and technical.

Let  $(\mathcal{V}_n)_{n < \omega}$  be an increasing (" $\subset$ ") sequence of monotone sequential contours, such that their ranks  $r(\mathcal{V}_n)$  are smaller than  $\gamma$  but converging to  $\gamma$ . For each  $n < \omega$  denote by  $V_n$  a (fixed) monotone sequential cascade such that  $\int V_n = \mathcal{V}_n$ . Let  $\{\mathcal{W}_\alpha, \alpha < \omega_1\}$  be an enumeration of all monotone sequential contours of rank  $\gamma$ . Let  $\omega^\omega = \{f_\alpha : \alpha < \omega_1\}$ .

By transfinite induction, for  $\alpha < \omega_1$  we build filters  $\mathcal{F}_\alpha$  together with their decreasing basis  $(F_\alpha^n)_{n < \omega}$ , such that:

1.  $\mathcal{F}_0$  is a Frechet filter;
2. for each  $\alpha < \omega_1$   $(F_\alpha^n)_{n < \omega}$  is a strictly decreasing base of  $\mathcal{F}_\alpha$ ;
3.  $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$  for  $\alpha < \beta$ ;
4.  $\mathcal{F}_\alpha = \bigcup_{\beta < \alpha} \mathcal{F}_\beta$  for  $\alpha$  limit ordinal;
5.  $\bigcup_{i < \omega} \mathcal{V}_i \cup \bigcup_{\alpha < \omega_1} \mathcal{F}_\alpha$  has finite intersection property;
6. for each  $\alpha < \omega_1$  there is  $F \in \mathcal{F}_{\alpha+1}$  such that  $f_\alpha[F] \in \mathcal{I}$ ;
7. for each  $\alpha < \omega_1$  there is  $F \in \mathcal{F}_{\alpha+1}$  such that the complement of  $F$  belongs to  $\mathcal{W}_\alpha$ .

Suppose that  $\mathcal{F}_\alpha$  is already define, we will show how to build  $\mathcal{F}_{\alpha+1}$ . This shall be done in five substeps. First for each  $\mathcal{V}_n$  and each  $F_\alpha^i$  we shall find  $H_{n,i}$  such that  $\mathcal{V}_n \cup \{F_\alpha^i, H_{n,i}\}$  has finite intersection property and  $f_\alpha[H_{n,i}] \in \mathcal{I}$ . Next we shall replace all the sets  $H_{n,i}$  by one set  $H_n$  such that  $\mathcal{V}_n \cup \mathcal{F}_\alpha \cup \{H_n\}$  has finite intersection property and  $f_\alpha[H_n] \in \mathcal{I}$ . On the third step one has to replace all the sets  $H_n$  by one set  $G_\alpha$  such that  $\bigcup_{m < \omega} \mathcal{V}_m \cup \mathcal{F}_\alpha \cup \{G_\alpha\}$  has finite intersection property and  $f_\alpha[G_\alpha] \in \mathcal{I}$ . The set  $G_\alpha$  take care on all the contours  $\mathcal{V}_n$ . Adding it as generator to  $F_{\alpha+1}$  will ensure preservation of conditions 5 and 6. On the fourth step will take care on condition 7 by adding set  $A_\alpha$  to the list of generators of  $F_{\alpha+1}$ . The last thing is to define decreasing base of a filter  $\mathcal{F}_{\alpha+1}$  and a filter itself.

*Substep i)* Fix  $n$  and  $i$ . Let us introduce an axillary definition.

*Definition:* Fix a monotone sequential cascade  $V$ , a set  $F$  and a function  $f \in \omega^\omega$ . For each  $v \in V$ , we write  $U \in \mathbf{S}(v)$  if

1.  $U \subset \max v^\uparrow$ ;



2.  $(U \cap F) \# \int v^\uparrow$ ;
3.  $\text{card}(f[U \cap F_\alpha^i]) = 1$ .

We following claim is crucial:

**Proposition 3.2.** *One that one of the following possibilities holds:*

A)  $\mathbf{S}(\emptyset_V) \neq \emptyset$ ;

B) *there is an antichain (with respect to the order of a cascade)  $\mathbb{A} \subset V$  such that:*

1.  $\mathbf{S}(v) = \emptyset$  for all  $v \in \mathbb{A}$ ,
2.  $(\bigcup \{\max w^\uparrow : w \in v^+, \mathbf{S}(w) \neq \emptyset\}) \# \int v^\uparrow$  for all  $v \in \mathbb{A}$ ,
3.  $(\bigcup \{\max v^\uparrow : v \in \mathbb{A}\}) \# \int V$ .

*Proof of the proposition.* First notice, that in definition of  $\mathbf{S}$  one can replace cardinality one by finite in condition 3), and that the replacement do not influence non-emptiness of  $\mathbf{S}(v)$ .

The proof is inductive by the rank of cascade  $V$ .

*First step:*  $r(V) = 1$ . If case A holds, then we are done, so without loss of generality  $f(U \cap F)$  is infinite for each  $U \cap F \in \max V$  such that  $U \# \int V$ . But since  $r(V) = 1$ , thus  $\text{card}(f(\max w \cap F)) \leq 1$ , for each  $w \in v^+$ . And since  $F \# \int V$  thus

$$\left( \bigcup \{(\max w \cap F) : w \in v^+, \text{card}(f(\max w \cap F)) = 1\} \right) \# \int V,$$

We put  $\mathbb{A} = \{\emptyset_V\}$  and see that case B holds.

*Inductive step :* Suppose that the proposition is true for each  $\beta < \alpha < \omega_1$ . So take  $V$  that  $r(V) = \alpha$ . Again if case A holds, then we are done, so without loss of generality assume that  $f(U \cap F)$  is infinite for each  $U \cap F \subset \max V$  such that  $U \# \int V$ . By inductive assumption, for each successor  $w$  of  $\emptyset_V$  either case A holds for cascade  $w^\uparrow$  either case B holds for for cascade  $w^\uparrow$ .

Split the set  $\emptyset_V^+$  of immediate successors of  $\emptyset_V$  into to subsets:

$$V^A = \{w \in \emptyset_V : \text{case A holds}\}, \quad V^B = \{w \in \emptyset_V : \text{case B holds}\}.$$

Since  $F \# \int V$ , we have two possibilities:

$$\left( \bigcup_{w \in V^A} (\max w^\uparrow \cap F) \right) \# \int V \text{ or } \left( \bigcup_{w \in V^B} (\max w^\uparrow \cap F) \right) \# \int V.$$

In the first case  $\mathbb{A} = \{\emptyset_V\}$  we was looking for.

In the second case, for each  $w \in V^B$  there is a claimed (by inductive assumptions) antichain  $\mathbb{A}_w$  in  $w^\uparrow$ . Put  $\mathbb{A} = \bigcup_{w \in V^B} \mathbb{A}_w$ . This finishes proof of the proposition.  $\blacksquare$

We can come back to the main proof.

We apply a proposition to cascade  $V_n$ , set  $F_\alpha^i$  and a function  $f_\alpha$ . In the case A we take any  $U \in \mathbf{S}(\emptyset_{V_n})$  and denote it by  $H_{n,i}$ .

In the case B for any  $v \in \mathbb{A}$  we fix  $U_w \in \mathbf{S}(w)$  for every  $w \in v^+$  for which  $\mathbf{S}(w) \neq \emptyset$ ; for all the other  $w \in V_n$  let  $U_w = \emptyset$ . For  $v \in \mathbb{A}$  consider  $T_v = \bigcup_{w \in v^+} U_w$ , and notice that  $f_\alpha[T_v]$  is infinite. Since  $\mathcal{I}$  is tall there is an infinite  $I_v \in \mathcal{I}$  such that  $I_v \subset f_\alpha[T_v]$ . Since  $\mathcal{I}$  is a P-ideal, there is infinite  $I_{n,i} \in \mathcal{I}$  such that  $I_v \setminus I_{n,i}$  is finite for all  $v \in \mathbb{A}$ . Put  $H_{n,i} = f_\alpha^{-1}[I_{n,i}]$ .

*Substep ii)* Now we will show how to replace sets  $H_{n,i}$  by one set  $H_n$ . Consider two possibilities:

C) there is an infinite  $K \subset \omega$  that  $f_\alpha[H_{n,i}]$  is infinite for each  $i \in K$ ;

D) there is an infinite  $K \subset \omega$  that  $f_\alpha[H_{n,i}]$  is a singleton for each  $i \in K$ .

In both cases since  $(F_\alpha^i)_{i < \omega}$  is decreasing, without loss of generality we may assume that  $K = \omega$ .

In the case C, since  $\mathcal{I}$  is a P-ideal, there is infinite  $I_n \in \mathcal{I}$  such that  $I_n \setminus I_{n,i}$  is finite for each  $i < \omega$ . Put  $H_n = f_\alpha^{-1}[I_n]$ .

In the case D we have two sub-cases:

If  $f_\alpha[\bigcup_{i < \omega} H_{n,i}]$  is infinite, then since  $\mathcal{I}$  is tall, there is an infinite  $I_n \in \mathcal{I}$  such that  $I_n \subset f_\alpha[\bigcup_{i < \omega} H_{n,i}]$ , and we put  $H_n = f_\alpha^{-1}[I_n]$ .

If  $f_\alpha[\bigcup_{i < \omega} H_{n,i}]$  is finite; then there is  $j \in f_\alpha[\bigcup_{i < \omega} H_{n,i}]$  that  $f_\alpha^{-1}[\{j\}] = H_{n,i}$  for infinite many  $i$ 's, and we put  $H_n = f_\alpha^{-1}[\{j\}]$ .

Clearly, in both cases  $\mathcal{V}_n \cup \mathcal{F}_\alpha \cup \{H_n\}$  has finite intersection property and  $f_\alpha[H_n] \in \mathcal{I}$ .

*Substep iii)* On this step we have to find set  $G_\alpha$  which can replace each  $H_n$ . We have shown that for each  $n$  there is a set  $H_n$  such that  $f_\alpha[H_n] \in \mathcal{I}$ . In fact we got a little bit more: either  $f_\alpha[H_n]$  is infinite but belongs to  $\mathcal{I}$ ,

either  $f_\alpha[H_n]$  is a singleton. We set

$$S = \{n < \omega : (\exists R_n) : \mathcal{V}_n \cup \mathcal{F}_\alpha \cup \{R_n\} \text{ has f.i.p. and } f_\alpha[R_n] \text{ is singleton} \}$$

It could happen that  $f_\alpha[H_n]$  is infinite but  $n \in S$  and for some  $R_n$  as above an image  $f_\alpha[R_n]$  is singleton. In this case we replace  $H_n$  by any  $R_n$ . For  $n \in \omega \setminus S$  we leave  $H_n$  unchanged. Once again proof splits into two cases: either  $S$  is infinite, either it is finite.

For infinite  $S$ : Without loss of generality (since  $(\mathcal{V}_n)$  is increasing) we may assume that  $S = \omega$  i.e.  $f_\alpha[H_n]$  is a singleton for each  $n < \omega$ .

If  $f_\alpha[\bigcup_{n < \omega} H_n]$  is finite, then there is  $j \in f_\alpha[\bigcup_{n < \omega} H_n]$  such that  $f_\alpha[H_n] = \{j\}$  for infinite many  $n$ . Since  $\mathcal{V}_n$  is increasing and  $(F_\alpha^n)$  is decreasing, a family  $\bigcup_{n < \omega} \mathcal{V}_n \cup \mathcal{F}_\alpha \cup f_\alpha^{-1}[\{j\}]$  has finite intersection property. Put  $G_\alpha = f_\alpha^{-1}[\{j\}]$ .

If  $f_\alpha[\bigcup_{n < \omega} H_n]$  is infinite, then, since  $\mathcal{I}$  is tall, there is infinite  $I_\alpha \in \mathcal{I}$  such that  $I_\alpha \subset f_\alpha[\bigcup_{n < \omega} H_n]$ . Since  $\mathcal{V}_n$  is increasing and  $(F_\alpha^n)$  is decreasing, a family  $\bigcup_{n < \omega} \mathcal{V}_n \cup \mathcal{F}_\alpha \cup f_\alpha^{-1}[I_\alpha]$  has finite intersection property. Put  $G_\alpha = f_\alpha^{-1}[I_\alpha]$ .

For finite  $S$ : Without loss of generality (since  $(\mathcal{V}_n)$  is increasing) we may assume that  $S = \emptyset$  i.e.  $f_\alpha[H_n]$  is infinite for each  $n < \omega$ .

Since  $\mathcal{I}$  is a P-ideal, and  $f_\alpha[H_n] \in \mathcal{I}$ , there is  $I_\alpha \in \mathcal{I}$  such that  $f_\alpha[H_n] \setminus I_\alpha$  is finite for each  $n < \omega$ .

Since the sequence  $(\mathcal{V}_n)$  is increasing, we have two possibilities: either  $f_\alpha^{-1}[I_\alpha] \# \mathcal{V}_n$  for all  $n < \omega$ ; either  $\neg f_\alpha^{-1}[I_\alpha] \# \mathcal{V}_n$  for almost every  $n$ . The second possibility cannot happen by the definition of  $\omega \setminus S$ . Put  $G_\alpha = f_\alpha^{-1}[I_\alpha]$ . It is easy to see that a family  $\bigcup_{n < \omega} \mathcal{V}_n \cup \mathcal{F}_\alpha \cup \{G_\alpha\}$  has finite intersection property.

*Substep iv)* Since the family  $\mathcal{F}_\alpha \cup \{G_\alpha\}$  is countable, thus by Lemma 2.2 there exists  $A_\alpha$  residual for the contour  $\mathcal{W}_\alpha$  and such that a family  $\bigcup_{n < \omega} \mathcal{V}_n \cup \mathcal{F}_\alpha \cup \{G_\alpha, A_\alpha\}$  has finite intersection property.

*Substep v)* Order  $\mathcal{F}_\alpha \cup \{G_\alpha\} \cup \{A_\alpha\}$  in type  $\omega$ , obtaining a sequence  $(\tilde{F}_\alpha^n : n < \omega)$ . Put  $F_\alpha^n = \bigcap_{m \leq n} \tilde{F}_\alpha^m$  to get decreasing sequence and let  $\mathcal{F}_{\alpha+1} = \langle \{F_{\alpha+1}^n : n < \omega\} \rangle$ .

Take any ultrafilter  $u$  that extends  $\bigcup_{n < \omega} \mathcal{V}_n \cup \bigcup_{\alpha < \omega_1} \mathcal{F}_\alpha$ . By condition 5)  $u$

is an  $\mathcal{I}$ -ultrafilter, by condition 6)  $u$  do not contain any monotone sequential contour of rank  $\gamma$  which jointly with  $\bigcup \mathcal{V}_n \subset u$  give us  $u \in \mathcal{P}_\gamma$ .

So the proof is done also for limit  $\gamma$ .

**Step 4:**  $\gamma = \omega_1$ . We will show a little more i.e. that there is a supercontour which is an  $\mathcal{I}$ -ultrafilter.

Again we list  ${}^\omega\omega = \{f_\alpha : \alpha < \omega_1\}$ , and we also list all pair (set and its complement) in the  $\omega_1$ -sequence of pairs  $(A_\alpha, \omega \setminus A_\alpha)$  that way that each set appears in the sequence only ones: or a set  $A_\alpha$  or as complement  $\omega \setminus A_\alpha$ .

We will build an  $\omega_1$  sequence  $(V_\alpha : \alpha < \omega_1)$  of monotone sequential cascades such that

1.  $\int V_\beta \subset \int V_\alpha$  for each  $\beta < \alpha < \omega_1$ .
2.  $r(V_\alpha) = \alpha$  for every  $\alpha < \omega_1$  ;
3.  $\max v_\alpha = \omega$  for every  $\alpha < \omega_1$ ;
4. there exist  $U \in \int V_{\alpha+1}$  such that  $f_\alpha[U] \in \mathcal{I}$
5.  $A_\alpha \in \int V_{\alpha+1}$  or  $\omega \setminus A_\alpha \in \int V_{\alpha+1}$ .

Define  $V_1$  as an arbitrary (fixed) monotone sequential cascade of rank 1. Suppose that we already defined cascades  $V_\beta$  for all  $\beta < \alpha < \omega_1$ .

Case 1)  $\alpha = \beta + 1$  is a successor. Take  $V_\beta$ , by step 3 there is a set  $H_\alpha$  such that  $H_\alpha \# \int V_\beta$  and  $f_\beta[H_\alpha] \in \mathcal{I}$ . Consider a cascade  $V_\beta^{\downarrow H_\alpha}$ ; this is a monotone sequential cascade of rank  $\beta$ . By the proof of Theorem 4.6 from [8] there is a monotone sequential cascade  $\tilde{V}_\alpha$  of rank  $\alpha$  such that  $\int V_\beta^{\downarrow H_\alpha} \subset \int \tilde{V}_\alpha$ . At least one of the elements of a pair  $(A_\alpha, \omega \setminus A_\alpha)$  meshes  $\int \tilde{V}_\alpha$ , denote it by  $B_\alpha$ . Now let  $V_\alpha = \tilde{V}_\alpha^{\downarrow B_\alpha}$ .

Case 2)  $\alpha$  is limit. Let  $V_\alpha$  be any monotone sequential cascade of rank  $\alpha$  such that  $\int V_\beta \subset \int V_\alpha$  for each  $\beta < \alpha$ . Such a cascade was constructed in the proof of Theorem 4.6 in [8].

Now it suffice to take  $u = \bigcup_{\alpha < \omega_1} \int V_\alpha$ . By construction  $u$  has a finite intersection property and is a supercontour, by 4)  $u$  is an ultrafilter and by 3)  $u$  is an  $\mathcal{I}$ -ultrafilter. ■

The assumption that an ideal  $\mathcal{I}$  is tall is essential: Flašková has proved in [9] Proposition 2.2, that if  $\mathcal{I}$  is not tall, then there is no  $\mathcal{I}$ -ultrafilters. One can easily see, that an ideal  $\mathcal{I}$  has to contains all singletons, also.

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# How high can Baumgartner's $\mathcal{I}$ -ultrafilters lie in the P-hierarchy ?

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## Abstract

Under CH we prove that for any tall P-ideal  $\mathcal{I}$  on  $\omega$  and for any ordinal  $\gamma \leq \omega_1$  there is an  $\mathcal{I}$ -ultrafilter (in the sense of Baumgartner), which belongs to the class  $\mathcal{P}_\gamma$  of P-hierarchy of ultrafilters. Since the class of  $\mathcal{P}_2$  ultrafilters coincides with a class of P-points, our result generalizes theorem of Flašková, which states that there are  $\mathcal{I}$ -ultrafilters which are not P-points.

## 1 Introduction

Baumgartner in the article *Ultrafilters on  $\omega$*  ([1]) introduced a notion of  $\mathcal{I}$ -ultrafilters:

Let  $\mathcal{I}$  be an ideal on  $\omega$ . A filter on  $\omega$  is an  $\mathcal{I}$ -ultrafilter, if and only if, for every function  $f \in \omega^\omega$  there is a set  $U \in \mathcal{u}$  such that  $f[U] \in \mathcal{I}$ .

This kind of ultrafilters was studied by large group of mathematicians. We shall mention only the most important papers in this subject from our point of view: J. Brendle [3], C. Laflamme [17], Shelah [20], [21], Błaszczyk [2]. The theory of  $\mathcal{I}$ -ultrafilters was developed by Flašková in a series of articles and in her Ph.D thesis [10].

In [10] Flašková proved under CH that for every tall P-ideal  $\mathcal{I}$  that contains all singletons, there is an  $\mathcal{I}$ -ultrafilter, which is not a P-point. Later she succeeded to replace CH by the assumption  $\mathfrak{p} = \mathfrak{c}$  [9].

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Ultrafilters on  $\omega$  may be classified with respect to sequential contours of different ranks, that is, iterations of the Fréchet filter by contour operations. This way an  $\omega_1$ -sequence  $\{\mathcal{P}_\alpha\}_{1 \leq \alpha \leq \omega_1}$  of pairwise disjoint classes of ultrafilters - the P-hierarchy - is obtained, where P-points correspond to the class  $\mathcal{P}_2$ , allowing us to look at the P-hierarchy as the extension of notion of P-point. The following theorem was proved by Starosolski, see [23] Proposition 2.1:

**Proposition 1.1.** *An ultrafilter  $u$  is a P-point if and only if  $u$  belongs to the class  $\mathcal{P}_2$  in P-hierarchy.*

Many important information about P-hierarchy may be found in [23]. For additional information regarding sequential cascades and contours one can look at [7], [8], [6], [22]. However the most important definitions and conventions shall be repeated below.

Since P-point correspond to  $\mathcal{P}_2$  ultrafilter in P-hierarchy of ultrafilters (more about P-hierarchy one can find below), it would be interesting to know to which classes of P-hierarchy can belong  $\mathcal{I}$ -ultrafilters. In this paper we shall show that it can be any class  $\mathcal{P}_\alpha$ . Let us introduce all necessary definitions and tools.

The set of natural numbers (finite ordinal numbers) we denote  $\omega$ . The filter considered in this paper will be defined on infinite countable set (except one indicated case). This will be usually a set  $\max V$  of maximal elements of a cascade  $V$  (see definition of cascade below) and we will often identify it with  $\omega$  without indication. The following convention will be applied without mentioning it:

*Conventions:* If  $u$  is a filter on  $A \subset B$ , then we identify  $u$  with the filter on  $B$  for which  $u$  is a filter-base. In particular we identify principal ultrafilter on  $\{v\}$  with principal ultrafilter generated on  $\omega$  by  $v$ . If  $\mathcal{F}$  is a filter base, then by  $\langle \mathcal{F} \rangle$  we denote a filter generated by  $\mathcal{F}$ .

The *cascade* is a tree  $V$  without infinite branches and with a least element  $\emptyset_V$ . A cascade is *sequential* if for each non-maximal element of  $V$  ( $v \in V \setminus \max V$ ) the set  $v^{+V}$  of immediate successors of  $v$  (in  $V$ ) is countably infinite. We write  $v^+$  instead of  $v^{+W}$  if it is known in which cascade the successors of  $v$  are considered. If  $v \in V \setminus \max V$ , then the set  $v^+$  (if infinite) may be endowed with an order of the type  $\omega$ , and then by  $(v_n)_{n \in \omega}$  we denote the sequence of elements of  $v^+$ , and by  $v_{nW}$  - the  $n$ -th element of  $v^{+W}$ .

The *rank* of  $v \in V$  ( $r_V(v)$  or  $r(v)$ ) is defined inductively as follows:  $r(v) = 0$  if  $v \in \max V$ , and otherwise  $r(v)$  is the least ordinal greater than the



ranks of all immediate successors of  $v$ . The rank  $r(V)$  of the cascade  $V$  is, by definition, the rank of  $\emptyset_V$ . If it is possible to order all sets  $v^+$  (for  $v \in V \setminus \max V$ ) so that for each  $v \in V \setminus \max V$  the sequence  $(r(v_n)_{n < \omega})$  is non-decreasing, then the cascade  $V$  is *monotone*, and we fix such an order on  $V$  without indication.

For  $v \in V$  we denote by  $v^\uparrow$  a subcascade of  $V$  built by  $v$  and all successors of  $v$ . We write  $v^\uparrow$  instead of  $v^{\uparrow V}$  if we know in which cascade the subcascade is included.

One may assume that cascade  $V$  is a family of subset of infinite countable set  $(\omega)$  and the order on  $V$  is inclusion. Indeed cascade  $V$  is isomorphic to a cascade  $\bar{V}$  such that:

- $\emptyset_{\bar{V}} = \omega$ ;
- $\bar{v}^+$  is a partition of  $\bar{v}$  for every  $\bar{v} \in \bar{V}$ :  
 $\bar{v} = \bigcup \{\bar{w} : \bar{w} \in \bar{v}^+\}$  and elements of  $\bar{v}^+$  are disjoint.
- $\bar{v}$  is singleton for every  $\bar{v} \in \max \bar{V}$ .

An isomorphism  $\bar{\cdot} : V \rightarrow \bar{V}$  is given by formula  $\bar{v} = \max v^\uparrow$ .

If  $\mathbb{F} = \{\mathcal{F}_s : s \in S\}$  is a family of filters on  $X$  and if  $\mathcal{G}$  is a filter on  $S$ , then the *contour of  $\{\mathcal{F}_s\}$  along  $\mathcal{G}$*  is defined by

$$\int_{\mathcal{G}} \mathbb{F} = \int_{\mathcal{G}} \{\mathcal{F}_s : s \in S\} = \bigcup_{G \in \mathcal{G}} \bigcap_{s \in G} \mathcal{F}_s.$$

Such a construction has been used by many authors ([11], [12], [13]) and is also known as a sum (or as a limit) of filters.

Operation of sum of filters we apply to define *contour of cascade*: Fix a cascade  $V$ . Let  $\mathcal{G}(v)$  be a filter on  $v^+$  for every  $v \in V \setminus \max V$ . For  $v \in \max V$  let  $\mathcal{G}(v)$  be a trivial ultrafilter on a singleton  $\{v\}$  (we can treat it as principal ultrafilter on  $\max v$  according to convention we assumed). This way we have defined a function  $v \mapsto \mathcal{G}(v)$ . We define contour of every sub-cascade  $v^\uparrow$  inductively with respect to rank of  $v$ :

$$\int^{\mathcal{G}} v^\uparrow = \{\{v\}\}$$

for  $v \in \max V$  (i.e.  $\int^{\mathcal{G}} v^\uparrow$  is just a trivial ultrafilter on singleton  $\{v\}$ ) ;

$$\int^{\mathcal{G}} v^\uparrow = \int_{\mathcal{G}(v)} \left\{ \int^{\mathcal{G}} w^\uparrow : w \in v^+ \right\}$$

for  $v \in V \setminus \max v$ . Eventually we put

$$\int^{\mathcal{G}} V = \int^{\mathcal{G}} \emptyset_V.$$

Usually we shall assume that all the filters  $\mathcal{G}(v)$  are Frechet (for  $v \in V \setminus \max V$ ). In that case we shall write  $\int V$  instead of  $\int^{\mathcal{G}} V$ .

Filters defined similar way were considered in [14], [15], [5], also.

Let  $V$  be a monotone sequential cascade and let  $u = \int V$ . Then a *rank*  $r(u)$  of  $u$  is, by definition, the rank of  $V$ .

It was shown in [8] that if  $\int V = \int W$ , then  $r(V) = r(W)$ .

We shall say that a set  $F$  *meshes* a contour  $\mathcal{V}$  ( $F \# \mathcal{V}$ ) if and only if  $\mathcal{V} \cup \{F\}$  has finite intersection property i.e can be extended to a filter. If  $\omega \setminus F \in \mathcal{V}$ , then we say that  $F$  is *residual* with respect to  $\mathcal{V}$ .

Let us define  $\mathcal{P}_\alpha$  for  $1 \leq \alpha < \omega_1$  on  $\beta\omega$  (see [23]) as follows:  $u \in \mathcal{P}_\alpha$  if there is no monotone sequential contour  $C_\alpha$  of rank  $\alpha$  such that  $C_\alpha \subset u$ , and for each  $\beta$  in the range  $1 \leq \beta < \alpha$  there exists a monotone sequential contour  $C_\beta$  of rank  $\beta$  such that  $C_\beta \subset u$ . Moreover, if for each  $\alpha < \omega_1$  there exists a monotone sequential contour  $C_\alpha$  of rank  $\alpha$  such that  $C_\alpha \subset u$ , then we write  $u \in \mathcal{P}_{\omega_1}$ .

Let us consider a monotone cascade  $V$  and a monotone sequential cascade  $W$ . We will say that  $W$  is a sequential extension of  $V$  if:

- 1)  $V$  is a subcascade of cascade  $W$ ,
- 2) if  $v^{+V}$  is infinite, then  $v^{+V} = v^{+W}$ ,
- 3)  $r_V(v) = r_W(v)$  for each  $v \in V$ .

Obviously, a monotone cascade may have many sequential extensions.

Notice that if  $W$  is a sequential extension of  $V$  and  $U \subset \max V$ , then  $U$  is residual for  $V$  if and only if  $U$  is residual for  $W$ .

It cannot be proven in ZFC that all the classes  $\mathcal{P}_\alpha$  are nonempty. The following theorem was proved in [23] Theorem 2.8:

**Theorem 1.2.** *The following statements are equivalent:*

1. *P-points exist,*

2.  $\mathcal{P}_\alpha$  classes are non-empty for each countable successor  $\alpha$ ,
3. There exists a countable successor  $\alpha > 1$  such that the class  $\mathcal{P}_\alpha$  is non-empty.

Starosolski has proved in [25] Theorem 6.7 that:

**Theorem 1.3.** *Assuming CH every class  $\mathcal{P}_\alpha$  is nonempty*

The main theorem presented in this paper is on the one side an extension of Starosolski's result, but on the side based on it.

Let us consider another technical notion which one could call a "restriction of a cascade". Let  $V$  be a monotone sequential cascade and let a set  $H$  meshes the contour  $\int V$ . By  $V^{\downarrow H}$  we denote a biggest monotone sequential cascade such that  $V^{\downarrow H} \subset V$  and  $\max V^{\downarrow H} \subset H$ . It is easy to see that  $H \in \int V^{\downarrow H}$ .

## 2 Lemmas

The following lemmas will be used in the prove of a main theorem.

The first lemma is one of lemmas proved in [24] (see: Lemma 6.3 ):

**Lemma 2.1.** *Let  $\alpha < \omega_1$  be a limit ordinal and let  $(\mathcal{V}_n : n < \omega)$  be a sequence of monotone sequential contours such that  $r(\mathcal{V}_n) < r(\mathcal{V}_{n+1}) < \alpha$  for every  $n$  and that  $\bigcup_{n < \omega} \mathcal{V}_n$  has finite intersection property. Then there is no monotone sequential contour  $\mathcal{W}$  of rank  $\alpha$  such that  $\mathcal{W} \subset \langle \bigcup_{n < \omega} \mathcal{V}_n \rangle$ .*

Since the paper with a prove of the above lemma is not published yet, the authors decided to included a prove at the end of this paper in a appendix.

As a corollary we get:

**Lemma 2.2.** *Let  $\alpha < \omega_1$  be a limit ordinal, let  $(\mathcal{V}_n)_{n < \omega}$  be an increasing (" $\subset$ ") sequence of monotone sequential contours, such that  $r(\mathcal{V}_n) < \alpha$  and let  $\mathcal{F}$  be a countable family of sets such that  $\bigcup_{n < \omega} \mathcal{V}_n \cup \mathcal{F}$  has finite intersection property. Then  $\langle \bigcup_{n < \omega} \mathcal{V}_n \cup \mathcal{F} \rangle$  do not contain any monotone sequential contour of rank  $\alpha$ .*

*Proof:* Assume that  $\mathcal{F}$  is finite. Let  $\mathcal{W}_n = \{U \cap \bigcap \mathcal{F} : U \in \mathcal{V}_n\}$ . It is easy to see that  $\mathcal{W}_n$  is monotone sequential contour of the same rank as  $\mathcal{V}_n$ . Consider a sequence  $(\mathcal{W}_n)$ . By Lemma 2.1 the union  $(\mathcal{W}_n)$  do not contains contour of rank  $\alpha$ .

Assume that  $\mathcal{F}$  is infinite. Order  $\mathcal{F}$  in  $\omega$  type, obtaining a sequence  $(F_n)_{n < \omega}$ . Next put

$$\mathcal{W}_n = \{U \cap \bigcap_{i \leq n} F_i : U \in \mathcal{V}_n\}.$$

Consider a sequence  $(\mathcal{W}_n : n < \omega)$  and use again Lemma 2.1 to show that the union  $(\mathcal{W}_n : n < \omega)$  do not contains contour of rank  $\alpha$ . ■

The following lemma is a straightforward extension of the claim contained in the proof of [9] Theorem 3.2. and since a proof is almost identical to the quoted one, we left it to the reader.

**Lemma 2.3.** *Let  $\mathcal{I}$  be a tall  $P$ -ideal that contains all singletons, let  $\{U_n : n < \omega\}$  be a pairwise disjoint sequence of subsets of  $\omega$ , let  $\{u_n : n < \omega\}$  be a sequence of  $\mathcal{I}$ -ultrafilters such that  $U_n \in u_n$ , finally let  $v$  be another one  $\mathcal{I}$ -ultrafilter. Then  $\int_v \{u_n : n < \omega\}$  is a  $\mathcal{I}$ -ultrafilter.*

As immediate consequence we get

**Lemma 2.4.** *If  $V$  is monotone sequential cascade,  $\mathcal{G}(v)$  is an  $P$ -point and  $\mathcal{I}$ -ultrafilter for each  $v \in V \setminus \max V$  and  $\mathcal{G}(v)$  is a trivial ultrafilter on a singleton  $\{v\}$  for  $v \in \max V$ , then  $\int^{\mathcal{G}} V$  is an  $\mathcal{I}$ -ultrafilter.*

Similar lemma as above one can formulate for ultrafilters in certain class in  $P$ -hierarchy instead of  $\mathcal{I}$ -ultrafilters, see [23] Theorem 2.5:

**Theorem 2.5.** *Let  $\gamma$  be an ordinal. Let  $V$  be a monotone sequential cascade of rank  $\gamma$ , let  $G(v)$  be a principal ultrafilter on  $\{v\}$  for  $v \in \max V$ , and let  $G(v)$  be a  $P$ -point on  $v^+$  for  $v \in V \setminus \max V$ . Then  $\int^G V \in P_{\gamma+1}$ .*

**Corollary 2.6.** *If  $V$  is monotone sequential cascade,  $\mathcal{G}(v)$  is an ultrafilter from the class  $\mathcal{P}_\gamma$  for each  $v \in V \setminus \max V$  and  $\mathcal{G}(v)$  is a trivial ultrafilter on a singleton  $\{v\}$  for  $v \in \max V$ , then  $\int^{\mathcal{G}} V$  belongs to the class  $\mathcal{P}_\gamma$ .*

### 3 Main result

In this section we shall present main result of the paper.

**Theorem 3.1.** *(CH) Let  $\mathcal{I}$  be a tall  $P$ -ideal that contain all singletons, and let  $\gamma \leq \omega_1$  be an ordinal. Then there exists an  $\mathcal{I}$ -ultrafilter  $u$  which belongs to  $\mathcal{P}_\gamma$ .*

*Proof.* We shall split proof into five cases:  $\gamma = 1$ ,  $\gamma = 2$ ,  $\gamma > 2$  is a successor ordinal (the main step),  $\gamma < \omega_1$  is limit ordinal,  $\gamma = \omega_1$ .

**Step 0:**  $\gamma = 1$  is clear, image of singleton ( $\mathcal{P}_1$  is a class of principal ultrafilters) is a singleton, so belongs to  $\mathcal{I}$ .

**Step 1:** for  $\gamma = 2$ .

We order all contours of rank 2 and all functions  $\omega \rightarrow \omega$  in  $\omega_1$ -sequences  $(\mathcal{W}_\alpha)_{\alpha < \omega_1}$ ,  $(f_\alpha)_{\alpha < \omega_1}$  respectively. By transfinite induction, for  $\alpha < \omega_1$  we build countable generated filters  $\mathcal{F}_\alpha$  together with their decreasing basis  $(F_\alpha^n)_{n < \omega}$ , such that:

1.  $\mathcal{F}_0$  is a Frechet filter;
2. for each  $\alpha < \omega_1$ , the sequence  $(F_\alpha^n)_{n < \omega}$  is strictly decreasing base of  $\mathcal{F}_\alpha$ ;
3.  $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$  for  $\alpha < \beta$ ;
4.  $\mathcal{F}_\alpha = \bigcup_{\beta < \alpha} \mathcal{F}_\beta$  for  $\alpha$  limit ordinal;
5. for each  $\alpha < \omega_1$  there is  $F \in \mathcal{F}_{\alpha+1}$  such that  $f_\alpha[F] \in \mathcal{I}$ ;
6. for each  $\alpha < \omega_1$  there is  $F \in \mathcal{F}_{\alpha+1}$  such that a complement of  $F$  belongs to  $\mathcal{W}_\alpha$ .

Suppose that  $\mathcal{F}_\alpha$  is already define, we will show how to build  $\mathcal{F}_{\alpha+1}$ . Since  $F_\alpha^n$  is strictly decreasing one can pick  $x_n \in F_\alpha^n \setminus F_\alpha^{n+1}$  for every  $n < \omega$ . Put  $T = \{x_n : n < \omega\}$ . There are two possibilities:

If  $f_\alpha[T]$  is finite then there is  $j \in f_\alpha[T]$  such that a preimage  $f_\alpha^{-1}[j]$  intersect infinite many of  $F_\alpha^n \setminus F_\alpha^{n+1}$ . In this case put  $G = f_\alpha^{-1}[j]$ .

If  $f_\alpha[T]$  is infinite, then since  $\mathcal{I}$  is tall there is  $I \in \mathcal{I}$  such that  $I \subset f_\alpha[T]$ . This time put  $G = f_\alpha^{-1}[I]$ .

Notice that  $\{F_\alpha^n : n < \omega\} \cup \{G_\alpha\}$  has finite intersection property and is countable. A subbase of any sequential contour of rank 2 has cardinality at least  $\mathfrak{d} > \aleph_0$ , thus none of them one is contained in  $\{F_\alpha^n : n < \omega\} \cup \{G_\alpha\}$ . This means that there is a set  $A_\alpha$  such that its complement belongs to  $\mathcal{W}_\alpha$  and a family  $\{F_\alpha^n : n < \omega\} \cup \{G_\alpha, A_\alpha\}$  has finite intersection property. Order  $\{F_\alpha^n : n < \omega\} \cup \{G_\alpha\} \cup \{A_\alpha\}$  in  $\omega$  type, obtaining a sequence  $(\tilde{F}_{\alpha+1}^n : n < \omega)$ . Put  $F_{\alpha+1}^n = \bigcap_{m \leq n} \tilde{F}_{\alpha+1}^m$  to get decreasing sequence and let  $\mathcal{F}_{\alpha+1} = \langle \{F_{\alpha+1}^n : n < \omega\} \rangle$ .

Take any ultrafilter  $u$  that extends  $\bigcup_{\alpha < \omega_1} \mathcal{F}_\alpha$ . By condition 5)  $u$  is an  $\mathcal{I}$ -ultrafilter, by condition 6)  $u$  do not contain any monotone sequential contour

of rank 2. Since by condition 1)  $u$  contains a Frechet filter it is not principal. Thus  $u$  is a P-point. (Note that on this step we do not use an assumption, that  $\mathcal{I}$  is a P-ideal.)

**Step 2:**  $\gamma$  is an arbitrary successor ordinal such that  $2 < \gamma < \omega_1$ . Let  $V$  be an arbitrary monotone sequential cascade of rank  $\gamma - 1$ . Let  $V \in v \mapsto \mathcal{G}(v)$  be a function such that:

- 1)  $\mathcal{G}(v)$  is an P-point and  $\mathcal{I}$ -ultrafilter for each  $v \in V \setminus \max V$  (such ultrafilters exist by step 1)
- 2)  $\mathcal{G}(v)$  be a trivial ultrafilter on a singleton  $\{v\}$  for  $v \in \max V$ .

Lemma 2.5 guarantee that  $\int^{\mathcal{G}} V \in \mathcal{P}_\gamma$ , whilst Lemma 2.4 ensures us that  $\int^{\mathcal{G}} V$  is an  $\mathcal{I}$ -ultrafilter.

So we are done for successor  $\gamma$ .

**Step 3:** for limit  $\gamma < \omega_1$ . The proof in this case is based on the same idea as step 1, but it is more sophisticated and technical.

Let  $(\mathcal{V}_n)_{n < \omega}$  be an increasing (" $\subset$ ") sequence of monotone sequential contours, such that their ranks  $r(\mathcal{V}_n)$  are smaller than  $\gamma$  but converging to  $\gamma$ . For each  $n < \omega$  denote by  $V_n$  a (fixed) monotone sequential cascade such that  $\int V_n = \mathcal{V}_n$ . Let  $\{\mathcal{W}_\alpha, \alpha < \omega_1\}$  be an enumeration of all monotone sequential contours of rank  $\gamma$ . Let  $\omega^\omega = \{f_\alpha : \alpha < \omega_1\}$ .

By transfinite induction, for  $\alpha < \omega_1$  we build filters  $\mathcal{F}_\alpha$  together with their decreasing basis  $(F_\alpha^n)_{n < \omega}$ , such that:

1.  $\mathcal{F}_0$  is a Frechet filter;
2. for each  $\alpha < \omega_1$   $(F_\alpha^n)_{n < \omega}$  is a strictly decreasing base of  $\mathcal{F}_\alpha$ ;
3.  $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$  for  $\alpha < \beta$ ;
4.  $\mathcal{F}_\alpha = \bigcup_{\beta < \alpha} \mathcal{F}_\beta$  for  $\alpha$  limit ordinal;
5.  $\bigcup_{i < \omega} \mathcal{V}_i \cup \bigcup_{\alpha < \omega_1} \mathcal{F}_\alpha$  has finite intersection property;
6. for each  $\alpha < \omega_1$  there is  $F \in \mathcal{F}_{\alpha+1}$  such that  $f_\alpha[F] \in \mathcal{I}$ ;
7. for each  $\alpha < \omega_1$  there is  $F \in \mathcal{F}_{\alpha+1}$  such that the complement of  $F$  belongs to  $\mathcal{W}_\alpha$ .

Suppose that  $\mathcal{F}_\alpha$  is already defined, we will show how to build  $\mathcal{F}_{\alpha+1}$ . This shall be done in five substeps. First for each  $\mathcal{V}_n$  and each  $F_\alpha^i$  we shall find  $H_{n,i}$  such that  $\mathcal{V}_n \cup \{F_\alpha^i, H_{n,i}\}$  has finite intersection property and  $f_\alpha[H_{n,i}] \in \mathcal{I}$ . Next we shall replace all the sets  $H_{n,i}$  by one set  $H_n$  such that  $\mathcal{V}_n \cup \mathcal{F}_\alpha \cup \{H_n\}$

has finite intersection property and  $f_\alpha[H_n] \in \mathcal{I}$ . On the third step one has to replace all the sets  $H_n$  by one set  $G_\alpha$  such that  $\bigcup_{m < \omega} \mathcal{V}_n \cup \mathcal{F}_\alpha \cup \{G_\alpha\}$  has finite intersection property and  $f_\alpha[G_\alpha] \in \mathcal{I}$ . The set  $G_\alpha$  take care on all the contours  $\mathcal{V}_n$ . Adding it as generator to  $F_{\alpha+1}$  will ensure preservation of conditions 5 and 6. On the fourth step will take care on condition 7 by adding set  $A_\alpha$  to the list of generators of  $F_{\alpha+1}$ . The last thing is to define decreasing base of a filter  $\mathcal{F}_{\alpha+1}$  and a filter itself.

*Substep i)* Fix  $n$  and  $i$ . Let us introduce an axillary definition.

*Definition:* Fix a monotone sequential cascade  $V$ , a set  $F$  and a function  $f \in \omega^\omega$ . For each  $v \in V$ , we write  $U \in \mathbf{S}(v)$  if

1.  $U \subset \max v^\uparrow$ ;
2.  $(U \cap F) \# \int v^\uparrow$ ;
3.  $\text{card}(f[U \cap F]) = 1$ .

We following claim is crucial:

**Proposition 3.2.** *One that one of the following possibilities holds:*

A)  $\mathbf{S}(\emptyset_V) \neq \emptyset$ ;

B) *there is an antichain (with respect to the order of a cascade)  $\mathbb{A} \subset V$  such that:*

1.  $\mathbf{S}(v) = \emptyset$  for all  $v \in \mathbb{A}$ ,
2.  $(\bigcup \{\max w^\uparrow : w \in v^+, \mathbf{S}(w) \neq \emptyset\}) \# \int v^\uparrow$  for all  $v \in \mathbb{A}$ ,
3.  $(\bigcup \{\max v^\uparrow : v \in \mathbb{A}\}) \# \int V$ .

*Proof of the proposition.* First notice, that in definition of  $\mathbf{S}$  one can replace cardinality one by finite in condition 3), and that the replacement do not influence non-emptiness of  $\mathbf{S}(v)$ .

The proof is inductive by the rank of cascade  $V$ .

*First step:*  $r(V) = 1$ . If case A holds, then we are done, so without loss of generality  $f(U \cap F)$  is infinite for each  $U \cap F \in \max V$  such that  $U \# \int V$ . But since  $r(V) = 1$ , thus  $\text{card}(f(\max w \cap F)) \leq 1$ , for each  $w \in v^+$ . And since  $F \# \int V$  thus

$$\left( \bigcup \{(\max w \cap F) : w \in v^+, \text{card}(f(\max w \cap F)) = 1\} \right) \# \int V,$$

We put  $\mathbb{A} = \{\emptyset_V\}$  and see that case B holds.

*Inductive step :* Suppose that the proposition is true for each  $\beta < \alpha < \omega_1$ . So take  $V$  that  $r(V) = \alpha$ . Again if case A holds, then we are done, so without loss of generality assume that  $f(U \cap F)$  is infinite for each  $U \cap F \subset \max V$  such that  $U \# \int V$ . By inductive assumption, for each successor  $w$  of  $\emptyset_V$  either case A holds for cascade  $w^\uparrow$  either case B holds for for cascade  $w^\uparrow$ .

Split the set  $\emptyset_V^+$  of immediate sucesors of  $\emptyset_V$  into to subsets:

$$V^A = \{w \in \emptyset_V : \text{case A holds}\}, \quad V^B = \{w \in \emptyset_V : \text{case B holds}\}.$$

Since  $F \# \int V$ , we have two possibilities:

$$\left( \bigcup_{w \in V^A} (\max w^\uparrow \cap F) \right) \# \int V \text{ or } \left( \bigcup_{w \in V^B} (\max w^\uparrow \cap F) \right) \# \int V.$$

In the first case  $\mathbb{A} = \{\emptyset_V\}$  we was looking for.

In the second case, for each  $w \in V^B$  there is a claimed (by inductive assumptions) antichain  $\mathbb{A}_w$  in  $w^\uparrow$ . Put  $\mathbb{A} = \bigcup_{w \in V^B} \mathbb{A}_w$ . This finishes proof of the proposition.  $\blacksquare$

We can come back to the main proof.

We apply a proposition to cascade  $V_n$ , set  $F_\alpha^i$  and a function  $f_\alpha$ . In the case A we take any  $U \in \mathbf{S}(\emptyset_{V_n})$  and denote it by  $H_{n,i}$ .

In the case B for any  $v \in \mathbb{A}$  we fix  $U_w \in \mathbf{S}(w)$  for every  $w \in v^+$  for which  $\mathbf{S}(w) \neq \emptyset$ ; for all the other  $w \in V_n$  let  $U_w = \emptyset$ . For  $v \in \mathbb{A}$  consider  $T_v = \bigcup_{w \in v^+} U_w$ , and notice that  $f_\alpha[T_v]$  is infinite. Since  $\mathcal{I}$  is tall there is an infinite  $I_v \in \mathcal{I}$  such that  $I_v \subset f_\alpha[T_v]$ . Since  $\mathcal{I}$  is a P-ideal, there is infinite  $I_{n,i} \in \mathcal{I}$  such that  $I_v \setminus I_{n,i}$  is finite for all  $v \in \mathbb{A}$ . Put  $H_{n,i} = f_\alpha^{-1}[I_{n,i}]$ .

*Substep ii)* Now we will show how to replace sets  $H_{n,i}$  by one set  $H_n$ . Consider two possibilities:

C) there is an infinite  $K \subset \omega$  that  $f_\alpha[H_{n,i}]$  is infinite for each  $i \in K$ ;

D) there is an infinite  $K \subset \omega$  that  $f_\alpha[H_{n,i}]$  is a singleton for each  $i \in K$ .

In both cases since  $(F_\alpha^i)_{i < \omega}$  is decreasing, without loss of generality we may assume that  $K = \omega$ .

In the case C, since  $\mathcal{I}$  is a P-ideal, there is infinite  $I_n \in \mathcal{I}$  such that  $I_n \setminus I_{n,i}$  is finite for each  $i < \omega$ . Put  $H_n = f_\alpha^{-1}[I_n]$ .

In the case D we have two sub-cases:



If  $f_\alpha[\bigcup_{i<\omega} H_{n,i}]$  is infinite, then since  $\mathcal{I}$  is tall, there is an infinite  $I_n \in \mathcal{I}$  such that  $I_n \subset f_\alpha[\bigcup_{i<\omega} H_{n,i}]$ , and we put  $H_n = f_\alpha^{-1}[I_n]$ .

If  $f_\alpha[\bigcup_{i<\omega} H_{n,i}]$  is finite; then there is  $j \in f_\alpha[\bigcup_{i<\omega} H_{n,i}]$  that  $f_\alpha^{-1}[\{j\}] = H_{n,i}$  for infinite many  $i$ 's, and we put  $H_n = f_\alpha^{-1}[\{j\}]$ .

Clearly, in both cases  $\mathcal{V}_n \cup \mathcal{F}_\alpha \cup \{H_n\}$  has finite intersection property and  $f_\alpha[H_n] \in \mathcal{I}$ .

*Substep iii)* On this step we have to find set  $G_\alpha$  which can replace each  $H_n$ . We have shown that for each  $n$  there is a set  $H_n$  such that  $f_\alpha[H_n] \in \mathcal{I}$ . In fact we got a little bit more: either  $f_\alpha[H_n]$  is infinite but belongs to  $\mathcal{I}$ , either  $f_\alpha[H_n]$  is a singleton. We set

$$S = \{n < \omega : (\exists R_n) : \mathcal{V}_n \cup \mathcal{F}_\alpha \cup \{R_n\} \text{ has f.i.p. and } f_\alpha[R_n] \text{ is singleton} \}$$

It could happen that  $f_\alpha[H_n]$  is infinite but  $n \in S$  and for some  $R_n$  as above an image  $f_\alpha[R_n]$  is singleton. In this case we replace  $H_n$  by any  $R_n$ . For  $n \in \omega \setminus S$  we leave  $H_n$  unchanged. Once again proof splits into two cases: either  $S$  is infinite, either it is finite.

For infinite  $S$ : Without loss of generality (since  $(\mathcal{V}_n)$  is increasing) we may assume that  $S = \omega$  i.e.  $f_\alpha[H_n]$  is a singleton for each  $n < \omega$ .

If  $f_\alpha[\bigcup_{n<\omega} H_n]$  is finite, then there is  $j \in f_\alpha[\bigcup_{n<\omega} H_n]$  such that  $f_\alpha[H_n] = \{j\}$  for infinite many  $n$ . Since  $\mathcal{V}_n$  is increasing and  $(F_\alpha^n)$  is decreasing, a family  $\bigcup_{n<\omega} \mathcal{V}_n \cup \mathcal{F}_\alpha \cup f_\alpha^{-1}[\{j\}]$  has finite intersection property. Put  $G_\alpha = f_\alpha^{-1}[\{j\}]$ .

If  $f_\alpha[\bigcup_{n<\omega} H_n]$  is infinite, then, since  $\mathcal{I}$  is tall, there is infinite  $I_\alpha \in \mathcal{I}$  such that  $I_\alpha \subset f_\alpha[\bigcup_{n<\omega} H_n]$ . Since  $\mathcal{V}_n$  is increasing and  $(F_\alpha^n)$  is decreasing, a family  $\bigcup_{n<\omega} \mathcal{V}_n \cup \mathcal{F}_\alpha \cup f_\alpha^{-1}[I_\alpha]$  has finite intersection property. Put  $G_\alpha = f_\alpha^{-1}[I_\alpha]$ .

For finite  $S$ : Without loss of generality (since  $(\mathcal{V}_n)$  is increasing) we may assume that  $S = \emptyset$  i.e.  $f_\alpha[H_n]$  is infinite for each  $n < \omega$ .

Since  $\mathcal{I}$  is a P-ideal, and  $f_\alpha[H_n] \in \mathcal{I}$ , there is  $I_\alpha \in \mathcal{I}$  such that  $f_\alpha[H_n] \setminus I_\alpha$  is finite for each  $n < \omega$ .

Since the sequence  $(\mathcal{V}_n)$  is increasing, we have two possibilities: either  $f_\alpha^{-1}[I_\alpha] \# \mathcal{V}_n$  for all  $n < \omega$ ; either  $\neg f_\alpha^{-1}[I_\alpha] \# \mathcal{V}_n$  for almost every  $n$ . The second possibility cannot happen by the definition of  $\omega \setminus S$ . Put  $G_\alpha = f_\alpha^{-1}[I_\alpha]$ . It is easy to see that a family  $\bigcup_{n<\omega} \mathcal{V}_n \cup \mathcal{F}_\alpha \cup \{G_\alpha\}$  has finite intersection property.

*Substep iv)* Since the family  $\mathcal{F}_\alpha \cup \{G_\alpha\}$  is countable, thus by Lemma 2.2 there exists  $A_\alpha$  residual for the contour  $\mathcal{W}_\alpha$  and such that a family  $\bigcup_{n < \omega} \mathcal{V}_n \cup \mathcal{F}_\alpha \cup \{G_\alpha, A_\alpha\}$  has finite intersection property.

*Substep v)* Order  $\mathcal{F}_\alpha \cup \{G_\alpha\} \cup \{A_\alpha\}$  in type  $\omega$ , obtaining a sequence  $(\tilde{F}_\alpha^n : n < \omega)$ . Put  $F_\alpha^n = \bigcap_{m \leq n} \tilde{F}_\alpha^m$  to get decreasing sequence and let  $\mathcal{F}_{\alpha+1} = \langle \{F_{\alpha+1}^n : n < \omega\} \rangle$ .

Take any ultrafilter  $u$  that extends  $\bigcup_{n < \omega} \mathcal{V}_n \cup \bigcup_{\alpha < \omega_1} \mathcal{F}_\alpha$ . By condition 5)  $u$  is an  $\mathcal{I}$ -ultrafilter, by condition 6)  $u$  do not contain any monotone sequential contour of rank  $\gamma$  which jointly with  $\bigcup \mathcal{V}_n \subset u$  give us  $u \in \mathcal{P}_\gamma$ .

So the proof is done also for limit  $\gamma$ .

**Step 4:**  $\gamma = \omega_1$ . We will show a little more i.e. that there is a supercontour which is an  $\mathcal{I}$ -ultrafilter.

Again we list  ${}^\omega\omega = \{f_\alpha : \alpha < \omega_1\}$ , and we also list all pair (set and its complement) in the  $\omega_1$ -sequence of pairs  $(A_\alpha, \omega \setminus A_\alpha)$  that way that each set appears in the sequence only ones: or a set  $A_\alpha$  or as complement  $\omega \setminus A_\alpha$ .

We will build an  $\omega_1$  sequence  $(V_\alpha : \alpha < \omega_1)$  of monotone sequential cascades such that

1.  $\int V_\beta \subset \int V_\alpha$  for each  $\beta < \alpha < \omega_1$ .
2.  $r(V_\alpha) = \alpha$  for every  $\alpha < \omega_1$  ;
3.  $\max v_\alpha = \omega$  for every  $\alpha < \omega_1$ ;
4. there exist  $U \in \int V_{\alpha+1}$  such that  $f_\alpha[U] \in \mathcal{I}$
5.  $A_\alpha \in \int V_{\alpha+1}$  or  $\omega \setminus A_\alpha \in \int V_{\alpha+1}$ .

Define  $V_1$  as an arbitrary (fixed) monotone sequential cascade of rank 1. Suppose that we already defined cascades  $V_\beta$  for all  $\beta < \alpha < \omega_1$ .

Case 1)  $\alpha = \beta + 1$  is a successor. Take  $V_\beta$ , by step 3 there is a set  $H_\alpha$  such that  $H_\alpha \# \int V_\beta$  and  $f_\delta[H_\delta] \in \mathcal{I}$ . Consider a cascade  $V_\beta^{\downarrow H_\alpha}$ ; this is a monotone sequential cascade of rank  $\beta$ . By the proof of Theorem 4.6 from [8] there is a monotone sequential cascade  $\tilde{V}_\alpha$  of rank  $\alpha$  such that  $\int V_\beta^{\downarrow H_\alpha} \subset \int \tilde{V}_\alpha$ . At least one of the elements of a pair  $(A_\alpha, \omega \setminus A_\alpha)$  meshes  $\int \tilde{V}_\alpha$ , denote it by  $B_\alpha$ . Now let  $V_\alpha = \tilde{V}_\alpha^{\downarrow B_\alpha}$ .

Case 2)  $\alpha$  is limit. Let  $V_\alpha$  be any monotone sequential cascade of rank  $\alpha$  such that  $\int V_\beta \subset \int V_\alpha$  for each  $\beta < \alpha$ . Such a cascade was constructed in the proof of Theorem 4.6 in [8].

Now it suffice to take  $u = \bigcup_{\alpha < \omega_1} \int V_\alpha$ . By construction  $u$  has a finite intersection property and is a supercontour, by 4)  $u$  is an ultrafilter and by 3)  $u$  is an  $\mathcal{I}$ -ultrafilter. ■

The assumption that an ideal  $\mathcal{I}$  is tall is essential: Flašková has proved in [9] Proposition 2.2, that if  $\mathcal{I}$  is not tall, then there is no  $\mathcal{I}$ -ultrafilters. One can easily see, that an ideal  $\mathcal{I}$  has to contains all singletons, also.

## 4 Appendix

In this appendix we shall prove Lemma 2.1. The main tools we use is an operation of decreasing the rank of cascade described below. Let us introduce axillary notion:

Let  $V$  be a cascade and let  $x_0, x_1, x_2, \dots$  be immediate sucesors of  $\emptyset_V$ . We denote a sub-cascades  $x_i^\uparrow$  by  $V^{(i)}$ . Similarly for  $V^{(i)}$  if  $x_{i0}, x_{i1}, x_{i2}, \dots$  are immediate sucesors of  $\emptyset_{V^{(i)}} = x_i$  then we denote sub-cascades  $x_{ij}^\uparrow$  by  $V^{(i)(j)}$ .

We say that a cascade  $V$  is built by destruction of nods of rank 1 in a cascade  $W$  iff

- all elements of rank 1 are removed from  $W$ :  
 $V = W \setminus \{v \in W : r_W(v) = 1\};$
- immediate sucesors of elements which had rank 2 are sucesors of their former sucesors: if  $r_W(v) = 2$  then

$$v^{+V} = \bigcup \{w^{+W} : w \in v^{+W}\}.$$

Observe that if  $r(W)$  is finite then  $r(V) = r(W) - 1$ .

Assume that there we are given a cascade of rank  $\alpha$  and an ordinal  $\beta < \alpha$ . We shall describe a operation of decreasing of rank of a cascade  $W$ . The construction is inductive:

*$\alpha$  is finite:*

We can decrease rank of  $W$  from  $\alpha$  to  $\beta$  by applying  $\alpha - \beta$  times an operation of destructing nods of rank 1.

$\alpha$  is infinite:

$\beta = \bar{\beta} + 1$  is successor ordinal and we are able to decrease a rank of any cascade of rank smaller than  $\alpha$ . Let  $r(W) = \alpha$ . Consider cascades  $W^{(i)}$  for  $i < \omega$ . Of course  $r(W^{(i)}) < \alpha$  for every  $i$  and one can decrease their ranks to  $\bar{\beta}$ . Let  $V^{(i)}$  be cascades obtained from  $W^{(i)}$  by decreasing rank:  $r(V^{(i)}) = \bar{\beta}$ , and let  $V$  be a cascade obtained by gluing cascades  $V^{(i)}$  together. Thus  $r(V) = \bar{\beta} + 1 = \beta$ .

$\beta$  is limit ordinal and we are able to decrease a rank of any cascade of rank smaller than  $\alpha$ . Let  $r(W) = \alpha$ . Consider cascades  $W^{(i)}$  for  $i < \omega$  and a sequence of ordinal  $(\beta_i)_{i < \omega}$  increasing to  $\beta$ . Of course  $r(W^{(i)}) < \alpha$  for every  $i$  and one can decrease ranks of every  $W^{(i)}$  to  $\beta_i$ . Let  $V^{(i)}$  be cascades obtained from  $W^{(i)}$  by decreasing rank:  $r(V^{(i)}) = \beta_i$ , and let  $V$  be a cascade obtained by gluing cascades  $V^{(i)}$  together. Thus  $r(V) = \beta$ .

Observe that above described decreasing of rank of cascade  $W$  does not change  $\max W$ . If a cascade  $W$  is obtained from  $V$  by decreasing rank, then we write  $W \triangleleft V$ . Trivially  $V \triangleleft V$  for every  $V$ .

We shall make use of the following theorem (see: [6] ) :

**Theorem 4.1** (Dolecki). *If  $(\mathcal{V}_n)_{n < \omega}$  is a sequence of monotone sequential contours of rank less than  $\alpha$  and  $\bigcup_{n < \omega} \mathcal{V}_n$  has finite intersection property, then there is no monotone sequential contour  $\mathcal{W}$  of rank  $\alpha + 1$  such that  $\mathcal{W} \subset \langle \bigcup_{n < \omega} \mathcal{V}_n \rangle$ .*

Before we prove Lemma 2.1 we shall prove a following technical claim

**Lemma 4.2.** *Let  $V$  be a cascade of rank  $\alpha$ ,  $W$  be cascade obtained from  $V$  by decreasing rank of  $V$  to  $\beta < \alpha$  and let  $\beta < \gamma < \alpha$ . Then there is a cascade  $T$  of rank  $\gamma$  such that  $W \triangleleft T \triangleleft V$ .*

*Proof:* The proof is inductive on triples  $(\beta, \alpha, \gamma)$  where  $\beta \leq \gamma \leq \alpha$  and ordered lexicographically. Assume that for  $(\beta', \alpha', \gamma') < (\beta, \alpha, \gamma)$  lemma has been proved. For  $\gamma = \beta$  there is nothing to prove, so assume that  $\beta < \gamma$ .

Observe that if  $v \in W$  is an element of rank 1 in  $W$ , then its successors are maximal elements of cascade  $V$ :

$$v^{+W} = \max v^{\uparrow V}.$$

Consider two cases.

$\gamma$  is a limit ordinal: Denote by  $x_1, x_2, \dots$  succesors of  $\emptyset_V$ . Recall that  $\emptyset_V = \emptyset_W$  and  $\emptyset_V^+ = \emptyset_W^+$ . Fix increasing sequence  $(\gamma_n : n < \omega)$  converging to  $\gamma$  such that  $\gamma_n > r_V(x_n)$  for every  $n$ . We have

$$x_n^{\uparrow W} \triangleleft x_n^{\uparrow V} \text{ and } r(x_n^{\uparrow W}) < \beta, \quad r(x_n^{\uparrow V}) < \alpha.$$

Using inductive hypothesis one can find  $T^{(n)}$  such that  $r(T^{(n)}) = \gamma_n$  and

$$x_n^{\uparrow W} \triangleleft T^{(n)} \triangleleft x_n^{\uparrow V}.$$

Let  $T$  be obtained by gluing  $T^{(n)}$ .

$\gamma = \delta + 1$  is a succesor ordinal: Proceed similarly, by assume that a sequence  $(\gamma_n : n < \omega)$  is contantly equal  $\delta$ .  $\blacksquare$

Now we can turn out attention to the proof of Lemma 2.1.

*Proof of Lemma 2.1:*

Assume that there exists a contour  $\mathcal{W}$  of rank  $\alpha$  such that  $\mathcal{W} \subset \langle \bigcup_{n < \omega} \mathcal{V}_n \rangle$ . We build a cascade  $W$  and a sequence of cascades  $(W_n)_{n < \omega}$  such that:

- $\int W = \mathcal{W}$ ;
- $W_m \triangleleft W_{m+1}$  for every  $m$ ;
- $W_m \triangleleft W$  for every  $m$ ;
- $W_m$  is obtained by decreasing of rank of  $W$  (with cutted several branches the way not influencing contour) to  $\alpha_m + 3$ ;
- if  $r(W_m^{(i)}) = \alpha_m + 2$  then  $r(W_m^{(i)(j)}) = \alpha_m + 1$  for each  $j$ ;
- if  $r(W_m^{(i)}) < \alpha_m + 2$  then  $W_m^{(i)} = W_{m-1}^{(i)}$ .

Fix any cascade  $\bar{W}$  such that  $\int \bar{W} = \mathcal{W}$ . Let  $\bar{W}_m$  be a cascade obtained from  $\bar{W}$  by cutting every branch  $\bar{W}^{(i)}$  of rank smaller than  $\alpha_m + 2$  and every (sub-)branch  $\bar{W}^{(i)(j)}$  of rank smaller than  $\alpha_m + 1$ . Observe that we cut only finite many branches  $\bar{W}^{(i)}$  and for the other  $\bar{W}^{(i)}$  only finite many branches  $\bar{W}^{(i)(j)}$ . Thus  $\int \bar{W}_m = \int \bar{W} = \mathcal{W}$  for every  $m$ .

Let  $W_1$  be a cascade obtained from  $\bar{W}_1$  by decreasing ranks of  $W_1^{(i)(j)}$  to  $\alpha_1 + 1$  and let  $W = \bar{W}$ . Thus  $W_1 \triangleleft W$ . Assume that  $W_1 \triangleleft W_2 \triangleleft \dots \triangleleft W_m$  have been defined such that  $W_l \triangleleft W$  and  $r(W_l^{(i)(j)}) = \alpha_l + 1$  (thus  $r(W_l) = \alpha_l + 3$ )

for every  $l \leq m$ . We use Lemma 4.2 to cascades  $W_m^{(i)(j)}$  and  $W^{(i)(j)}$  to define  $W_{m+1}^{(i)(j)}$  of rank  $\alpha_{m+1} + 1$ . Gluing  $W_{m+1}^{(i)(j)}$  we obtain first  $W_{m+1}^{(i)}$  and next gluing them  $W_{m+1}$  such that  $W_m \triangleleft W_{m+1} \triangleleft W$  and  $r(W_{m+1}) = \alpha_{m+1} + 3$ .

Next we build a decreasing sequence  $(U_n)_{n < \omega}$  such that:

1.  $U_n \in \int W_n$ ;
2.  $U_n \notin \langle \bigcup_{i \leq n} \mathcal{V}_i \rangle$ ;
3.  $U_n \cap \max \bar{W}_{n+1} = U_{n+1} \cap \max \bar{W}_{n+1}$  for each  $n$ ;
4.  $U_n \cap \max W^{(k)} \in \int W^{(k)}$  for each  $k$ .

The last condition can follow difficulties in a construction. Therefore we shall replace in a construction cascades  $W_n$  by modified cascades  $\bar{W}_n$ , such that it will be not necessary to take care on condition (4). Fix  $n$ . We define cascade  $\bar{W}_n$  as follows: We remove from original  $W_n$  successors of  $\emptyset_{W_n}$  and we treat successors of removed elements as new successors of  $\emptyset_{\bar{W}_n} = \emptyset_{W_n}$ . Formally:

$$\emptyset_{\bar{W}_n}^+ = \bigcup \{w^+ : w \in \emptyset_{W_n}^+\}.$$

The rest of cascades we leaved unchanged. We denote obtained cascade  $\widetilde{W}_n$ .

Put  $U_0 = \omega$ . Assume that  $U_0, U_1, \dots, U_{n-1}$  was defined, but it is impossible to define  $U_n$ . This means that every set  $U \in \int \widetilde{W}_n$  is contained in  $\langle \bigcup_{i \leq n} \mathcal{V}_i \rangle$ . On the other site  $\max \widetilde{W}_n \in \mathcal{W}$  and so the family  $\{U \cap \max \bar{W}_n : U \in \bigcup_{i \leq n} \mathcal{V}_i\}$  has finite intersection property. By theorem of Dolecki  $\{U \cap \max \bar{W}_n : U \in \bigcup_{i \leq n} \mathcal{V}_i\}$  is not contained in the contour  $\int \bar{W}_n$  of rank  $\alpha_n + 2$ . A contradiction. On each step of induction we can put  $\bigcap_{i \leq n} U_i$  instead of  $U_n$  and assume that the sequence  $(U_n)_{n < \omega}$  is decreasing.

Let  $U = \bigcap_{n < \omega} U_n$ . Conditions (1)-(4) guarantee that  $U \in \int W$  and  $U \notin \langle \bigcup_{n < \omega} \mathcal{V}_n \rangle$ . Indeed, assume that  $U \in \langle \bigcup_{n < \omega} \mathcal{V}_n \rangle$ , then there is a finite  $M < \omega$  such that  $U \in \langle \bigcup_{n < M} \mathcal{V}_n \rangle$ . But  $U_M \notin \bigcup_{n \leq M} \mathcal{V}_n$  and  $U \subset U_M$ . Thus  $U \notin \langle \bigcup_{n < \omega} \mathcal{V}_n \rangle$ . A contradiction.

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